Applying Euler's Method to a second order ODE

A general second ordinary differential equation:

$$\frac{d^2x}{dt^2} + a(t,x)\frac{dx}{dt} + b(t,x)x = f(t,x) \quad (1)$$

can be converted to a first order system

$$\dot{\underline{X}} = \begin{bmatrix} 0 & 1 \\ -b(t,x) & -a(t,x) \end{bmatrix} \underline{X} + \underline{F} \quad (1)$$

where $\underline{X} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\dot{\underline{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$, $\underline{F} = \begin{bmatrix} 0 \\ f(t,x) \end{bmatrix}$ and $y = \frac{dx}{dt}$.

(See http://www.numerical-methods.com/ODE/system.pdf)

In any t-interval $t_{n-1} \leq t \leq t_n$ Euler's method advances the solution $\underline{X}$ from $\underline{X}_{n-1} \approx \underline{X}(t_{n-1})$ to $\underline{X}_n \approx \underline{X}(t_n)$ through approximating $\dot{\underline{X}} \approx \frac{\underline{X}_n - \underline{X}_{n-1}}{k}$.

Substituting this back into (1) gives

$$\frac{\underline{X}_n - \underline{X}_{n-1}}{k} = \begin{bmatrix} 0 & 1 \\ -b_{n-1} & -a_{n-1} \end{bmatrix} \underline{X}_{n-1} + \underline{F}_n ,$$

where $a_{n-1} = a(t_{n-1}, x_{n-1})$

Making $\underline{X}_n$ the subject gives the following relation:

$$\underline{X}_n = \underline{X}_{n-1} + k \begin{bmatrix} 0 & 1 \\ -b_{n-1} & -a_{n-1} \end{bmatrix} \underline{X}_{n-1} + k\underline{F}_n .$$

This may also be written as two simultaneous equations:

$$x_n = x_{n-1} + ky_{n-1}$$

$$y_n = y_{n-1} - kb_{n-1}x_{n-1} - ka_{n-1}y_{n-1} + kf_{n-1}$$

The equation (1) is demonstrated on a spreadsheet (Excel). The spreadsheets can be downloaded from the webpage http://www.numerical-methods.com/Euler