Applying Euler's Method to a second order ODE

A general second ordinary differential equation[1]:

$$\frac{d^2x}{dt^2}+a(t,x)\frac{dx}{dt}+b(t,x)x=f(t,x) \quad (1)$$

can be converted to a first order system

$$\underline{\dot{X}}=\begin{bmatrix}0 & 1\\ -b(t,x) & -a(t,x)\end{bmatrix}\underline{X}+\underline{F} \quad (1)$$

where $\underline{X}=\begin{bmatrix}x\\ y\end{bmatrix}$, $\underline{\dot{X}}=\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}$, $\underline{F}=\begin{bmatrix}0\\ f(t,x)\end{bmatrix}$ and $y=\frac{dx}{dt}$.

In any t-interval $t_{n-1} \le t \le t_n$ Euler's method advances the solution $\underline{X}$ from $\underline{X}_{n-1}\approx\underline{X}(t_{n-1})$ to $\underline{X}_n\approx\underline{X}(t_n)$ through approximating $\underline{\dot{X}}\approx\frac{\underline{X}_n-\underline{X}_{n-1}}{k}$.

Substituting this back into (1) gives

$$\frac{\underline{X}_n-\underline{X}_{n-1}}{k}=\begin{bmatrix}0 & 1\\ -b_{n-1} & -a_{n-1}\end{bmatrix}\underline{X}_{n-1}+\underline{F}_n ,$$

where $a_{n-1}=a(t_{n-1},x_{n-1})$

Making $\underline{X}_n$ the subject gives the following relation:

$$\underline{X}_n=\underline{X}_{n-1}+k\begin{bmatrix}0 & 1\\ -b_{n-1} & -a_{n-1}\end{bmatrix}\underline{X}_{n-1}+k\underline{F}_n .$$

This may also be written as two simultaneous equations:

$$x_n=x_{n-1}+ky_{n-1}$$

$$y_n=y_{n-1}-kb_{n-1}x_{n-1}-ka_{n-1}y_{n-1}+kf_{n-1}$$

The equation (1) is demonstrated on a spreadsheet (Excel)[2].

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[1] Ordinary Differential Equations
[2] Solving a second order ODE on Excel Spreadsheet