Runge-Kutta Methods

Runge Kutta methods are used to solve ordinary differential equations[1]. Runge-Kutta methods can be applied to a first order equation or to higher order ordinary differential equations through first resolving them to systems of first order equations. They are straightforward to apply and are generally much more efficient than the Euler method[2]. Runge-Kutta methods form a family of methods of varying order. Let us consider applying Runge-Kutta methods to the following first order ordinary differential equation:

$$\frac{dx}{dt} = f(t,x)$$

In any t-interval $t_{n-1} \le t \le t_n$ the Runge-Kutta method advances the solution x(t) from $x_{n-1} \approx x(t_{n-1})$ to $x_n \approx x(t_n)$. Two examples of Runge Kutta methods are given.

Second order Runge-Kutta method

$$k_1 = k\, f(t_{n-1}, x_{n-1})$$
$$k_2 = k\, f(t_{n-1}+\tfrac{1}{2}k,\ x_{n-1}+\tfrac{1}{2}k_1)$$
$$x_n = x_{n-1} + k_2$$

Fourth order Runge-Kutta method

$$k_1 = k\, f(t_{n-1}, x_{n-1})$$
$$k_2 = k\, f(t_{n-1}+\tfrac{1}{2}k,\ x_{n-1}+\tfrac{1}{2}k_1)$$
$$k_3 = k\, f(t_{n-1}+\tfrac{1}{2}k,\ x_{n-1}+\tfrac{1}{2}k_2)$$
$$k_4 = k\, f(t_{n-1}+k,\ x_{n-1}+k_3)$$
$$x_n = x_{n-1} + k_1/6 + k_2/3 + k_3/3 + k_4/6$$

This second order[3] and fourth order[4] methods are applied to the test problems below on Excel spreadsheets:

$$\frac{dx}{dt} = t \text{ with } x(0)=0,$$

$$\frac{dx}{dt} = x \text{ with } x(0)=1.$$

which have analytic solutions $x=t^2/2$ and $x=e^t$ respectively.

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[1] Ordinary Differential Equations
[2] Euler Method
[3] Excel spreadsheet demonstrating second order method
[4] Excel spreadsheet demonstrating fourth order method