Writing an second order ODE as a system of first order ODEs

A general second ordinary differential equation:

$$\frac{d^2x}{dt^2} + a(t,x)\frac{dx}{dt} + b(t,x)x = f(t,x)$$

can be converted to a first order system by the substitution $y = \frac{dx}{dt}$.

This results in a pair of equations:

$$\frac{dx}{dt} - y = 0,$$

$$\frac{dy}{dt} + a(t,x)y + b(t,x)x = f(t,x).$$

Using the notation $\dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$ we can write the above equations as follows:

$$\dot{x} - y = 0$$
$$\dot{y} + a(t,x)y + b(t,x)x = f(t,x)$$

Using the vector notation $\underline{X} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\dot{\underline{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$, $\underline{F} = \begin{bmatrix} 0 \\ f(t,x) \end{bmatrix}$ the above equations can be written as a matrix-vector equation as follows:

$$\dot{\underline{X}} = \begin{bmatrix} 0 & 1 \\ -b(t,x) & -a(t,x) \end{bmatrix} \underline{X} + \underline{F}$$

This is a system of first order ordinary differential equations.

The same technique can be applied to higher order systems. The dimension of the resulting matrices and vectors is equal to the order of the original ODE.